

Multi-point Distribution Function for the Continuous Time Random Walk

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We derive an explicit expression for the Fourier-Laplace transform of the two-point distribution function $p(x_1, t_1; x_2, t_2)$ of a continuous time random walk (CTRW), thus generalizing the result of Montroll and Weiss for the single point distribution function $p(x_1, t_1)$. The multi-point distribution function has a structure of a convolution of the Montroll-Weiss CTRW and the aging CTRW single point distribution functions. The correlation function $\langle x(t_1)x(t_2) \rangle$ for the biased CTRW process is found. The random walk foundation of the multi-time-space fractional diffusion equation [Baule and Friedrich [*Europhysics Letters* **77** 10002 (2007)]] is investigated using the unbiased CTRW in the continuum limit.

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I. INTRODUCTION

The continuous time random walk (CTRW) introduced in [1] more than forty years ago is a successful model for normal and anomalous diffusion in a variety of physical systems [2, 3, 4, 5, 6]. A new splash of interest in this old model was caused by the fact that it provides a firm statistical foundation of the fractional Fokker-Planck equation [7, 8], and is a simple model for the investigation of such intriguing phenomena of non-equilibrium statistical physics as weak ergodicity breaking [9] and statistical aging [10, 11, 12, 13]. The importance of CTRW as a minimal model of non-Markovian behavior is connected with its semi-Markovian (renewal) nature, which allows in many cases for an exact probabilistic description of the process of interest. Unlike Markovian diffusion processes, which are fully characterized by their transition probabilities, non-Markovian CTRW requires for the full hierarchy of multi-point distribution functions for its complete characterization [14, 15, 16, 17, 18]. Multi-point distribution functions and correlation functions are necessary tools to distinguish between CTRW stochastic dynamics from other non Markovian processes, e.g. fractional Brownian motion, as appearing in linear fracton models or in generalized Rouse models, see e.g. [19]. The simplest experimental way of discriminating these types of anomalous processes is based on different behavior of their correlation functions, as investigated e.g. in recent experiments on anomalous diffusion in single molecules [20, 21, 22]. Hence obviously it is a worthy goal to consider higher-order joint probability density functions (PDFs) and correlation functions of the CTRW.

In the present article we concentrate on decoupled CTRWs in the sub-diffusive and spatially homogeneous limit and obtain an exact expression for the two time characteristic function of the CTRW process in Laplace space. This main result is a generalization of the Montroll-Weiss equation [1] which gives the characteristic function of the single time PDF. We use our result to investigate the validity of a multi-point fractional kinetic equation of Baule and Friedrich [18]. We moreover obtain the two-point correlation function $\langle x_1(t_1)x_2(t_2) \rangle$ in a biased CTRW. We note that even this simple correlation function cannot be found from the Green's function $p(x_1, t_1)$ of the CTRW, since the process is non Markovian.

II. DEFINITIONS AND NOTATION

We consider a standard CTRW model in one dimension with a walker starting at the origin at time $t = 0$ [1, 2, 3, 4, 5, 6]. Waiting times between jumps are independent identically distributed (IID) random variables with a common PDF $\psi(t)$. After waiting the particle makes a jump and the process is then renewed. The jump lengths are also IID random variables, with a PDF $f(\delta x)$. The waiting times and the lengths of jumps are mutually independent.

Let $p(x_1, t_1; x_2, t_2) = p(x_1, t_1; x_2, t_2|0, 0)$ be the joint PDF of finding a walker at x_1 at time $t_1 > 0$ and at x_2 at time $t_2 > 0$. We concentrate first on the corresponding multi-point characteristic function, i.e. on the double Fourier double Laplace transform of $p(x_1, t_1; x_2, t_2)$:

$$p(k_1, \lambda_1; k_2, \lambda_2) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_0^{\infty} dt_1 \int_0^{\infty} dt_2 e^{ik_1 x_1 + ik_2 x_2 - \lambda_1 t_1 - \lambda_2 t_2} p(x_1, t_1; x_2, t_2). \quad (1)$$

All over the article, the original functions and their transforms can be distinguished on the ground of their variables $(x_1, t_1; x_2, t_2)$ and $(k_1, \lambda_1; k_2, \lambda_2)$ for originals and transforms, respectively. The same holds for other functions encountered in the text.

Let us now separate in Eq.(1) the contributions corre-

sponding to the cases $t_1 < t_2$ and $t_1 > t_2$ and rewrite the integral as a sum of the two terms:

$$p(k_1, \lambda_1; k_2, \lambda_2) = p_{<}(k_1, \lambda_1; k_2, \lambda_2) + p_{>}(k_1, \lambda_1; k_2, \lambda_2) \quad (2)$$

with

$$p_{<}(k_1, \lambda_1; k_2, \lambda_2) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 e^{ik_1 x_1 + ik_2 x_2} \int_0^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 e^{-\lambda_1 t_1 - \lambda_2 t_2} p(x_1, t_1; x_2, t_2) \quad (3)$$

$$p_{>}(k_1, \lambda_1; k_2, \lambda_2) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 e^{ik_1 x_1 + ik_2 x_2} \int_0^{\infty} dt_1 \int_0^{t_1} dt_2 e^{-\lambda_1 t_1 - \lambda_2 t_2} p(x_1, t_1; x_2, t_2). \quad (4)$$

For $t_1 < t_2$ we define the elapsed time $\tau = t_2 - t_1 > 0$ and the corresponding displacement $\Delta = x_2 - x_1$. Similarly, for $t_1 > t_2$ we take $\tau = t_1 - t_2$ and $\Delta = x_1 - x_2$. For $t_2 > t_1$ we define the joint PDF $g(x_1, t_1; \Delta, \tau)$ that the particle is at x_1 at time t_1 and then experiences a displacement of size Δ during the time interval τ . Similar joint PDF for $t_2 < t_1$ is denoted with $g(x_2, t_2; \Delta, \tau)$.

To obtain $p(k_1, \lambda_1; k_2, \lambda_2)$ we use $g(x_1, t_1; \Delta, \tau)$ and $g(x_2, t_2; \Delta, \tau)$ in the corresponding terms of Eq. (3) and Eq. (4). Changing in Eq. (3) the variables according to $t_2 = t_1 + \tau$ and $x_2 = x_1 + \Delta$ one gets

$$\begin{aligned} p_{<}(k_1, \lambda_1; k_2, \lambda_2) &= \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} d\Delta \int_0^{\infty} dt_1 \int_0^{\infty} d\tau e^{-\lambda_1 t_1 - \lambda_2(t_1 + \tau) + ik_1 x_1 + ik_2(x_1 + \Delta)} g(x_1, t_1; \Delta, \tau) \\ &= g(k_1 + k_2, \lambda_1 + \lambda_2; k_2, \lambda_2). \end{aligned} \quad (5)$$

A similar change of variables is made in Eq.(4) giving us

$$\begin{aligned} p(k_1, \lambda_1; k_2, \lambda_2) &= g(k_1 + k_2, \lambda_1 + \lambda_2; k_1, \lambda_1) \\ &+ g(k_1 + k_2, \lambda_1 + \lambda_2; k_2, \lambda_2). \end{aligned} \quad (6)$$

Thus, our next task is to find the characteristic function g . We consider $t_1 < t_2$ and soon concentrate on $g(k_1, \lambda_1; k, s)$ which is the double Laplace and double Fourier transform of $g(x_1, t_1; \Delta, \tau)$ according to the rule $x_1 \rightarrow k_1$, $t_1 \rightarrow \lambda_1$, $\Delta \rightarrow k$ and $\tau \rightarrow s$. Let n_1 be the random number of jumps made by a walker during the time interval $(0, t_1)$, n_2 the number of jumps made in the time interval $(t_1, t_1 + \tau)$, and $P_{n_1, n_2}(t_1, \tau)$ the joint probability of these random variables. Since waiting times and jump lengths are independent, we may write

$$g(x_1, t_1; \Delta, \tau) = \sum_{n_1, n_2=0}^{\infty} P_{n_1, n_2}(t_1, \tau) p(x_1; n_1) p(\Delta | x_1; n_2) \quad (7)$$

where $p(x_1; n_1)$ is the probability density to find a particle at x_1 after n_1 steps and $p(\Delta | x_1; n_2)$ is the probability density of the displacement Δ of a particle starting at x_1 after n_2 steps. Eq. (7) is the key to all our further considerations, and it shows that the problem can be divided into three: calculation of $P_{n_1, n_2}(t_1, \tau)$ which is not trivial, and the calculation of $p(x_1; n_1)$ and $p(\Delta | x_1; n_2)$. The latter two describe simple random walks in discrete time, moreover, since the jump lengths are independent $p(\Delta | x_1; n_2)$ does not depend directly on x_1 . In this case

the corresponding displacements

$$x_1 = \sum_{i=1}^{n_1} \delta x_i, \quad \Delta = \sum_{i=n_1+1}^{n_1+n_2} \delta x_i, \quad (8)$$

are sums of IID random variables, and their characteristic functions are:

$$\langle e^{ik_1 x_1} \rangle = \int_{-\infty}^{\infty} p(x_1; n_1) e^{ik_1 x_1} dx_1 = \langle e^{ik_1 \delta x} \rangle^{n_1} = f^{n_1}(k_1) \quad (9)$$

and

$$\langle e^{ik\Delta} \rangle = \int_{-\infty}^{\infty} p(\Delta | x_1; n_2) e^{ik\Delta} d\Delta = f^{n_2}(k) \quad (10)$$

where $f(k) = \langle \exp(ik\delta x) \rangle$ is the one step characteristic function i.e. the Fourier transform of $f(\delta x)$. Passing to the Fourier – Laplace representation we hence get

$$g(k_1, \lambda_1; k, s) = \sum_{n_1, n_2=0}^{\infty} P_{n_1, n_2}(\lambda_1, s) f^{n_1}(k_1) f^{n_2}(k). \quad (11)$$

where $P_{n_1, n_2}(\lambda_1, s)$ is the double Laplace transform of $P_{n_1, n_2}(t_1, \tau)$ in its temporal variables. We thus see that this Laplace-transform $P_{n_1, n_2}(\lambda_1, s)$ of the probability of the number of steps during the corresponding time intervals plays the key role in our theory and we turn now to determining it.

III. STATISTICS OF NUMBERS OF STEPS

$$P_{n_1, n_2}(\lambda_1, s)$$

Let us now consider the set of jump times as a point process on the time axis and let $\bar{t}_1, \bar{t}_2, \dots, \bar{t}_i, \dots$ denote the corresponding points: \bar{t}_1 is the instant of time when the first jump was made, \bar{t}_2 is the time when the second jump was made etc. As mentioned, according to the CTRW model the waiting times $\bar{t}_1, \bar{t}_2 - \bar{t}_1, \bar{t}_3 - \bar{t}_2$ etc are IID random variables with the common PDF $\psi(t)$. The realizations of the process with n_1 jumps up to time t_1 and n_2 jumps between t_1 and $t_1 + \tau$ are those that satisfy $\bar{t}_{n_1} < t_1 < \bar{t}_{n_1+1}$ and $\bar{t}_{n_1+n_2} < t_1 + \tau < \bar{t}_{n_1+n_2+1}$ respectively. We introduce the indicator function $I(x)$ of a logical variable x so that $I(x) = 1$ if the condition x in the brackets holds (x is true) and $I(x) = 0$ otherwise. As usual the probability for x to be true is then the mean $\langle I(x) \rangle$ of $I(x)$ over realizations. For the case under consideration

$$P_{n_1, n_2}(t_1, \tau) = \langle I(\bar{t}_{n_1} < t_1 < \bar{t}_{n_1+1}) I(\bar{t}_{n_1+n_2} < t_1 + \tau < \bar{t}_{n_1+n_2+1}) \rangle. \quad (12)$$

The argument of the second indicator function can be rewritten as $(\bar{t}_{n_1+n_2} - t_1 < \tau < \bar{t}_{n_1+n_2+1} - t_1)$. The double Laplace transform of $P_{n_1, n_2}(t_1, \tau)$ is

$$P_{n_1, n_2}(\lambda_1, s) = \int_0^\infty d\tau e^{-s\tau} \int_0^\infty dt_1 e^{-\lambda_1 t_1} P_{n_1, n_2}(t_1, \tau) = \left\langle \int_0^\infty dt_1 e^{-\lambda_1 t_1} I(\bar{t}_{n_1} < t_1 < \bar{t}_{n_1+1}) \times \right.$$

$$\left. \int_0^\infty d\tau e^{-s\tau} I(\bar{t}_{n_1+n_2} - t_1 < \tau < \bar{t}_{n_1+n_2+1} - t_1) \right\rangle \quad (13)$$

where we used the linearity of the Laplace transform to interchange the sequence of integration and ensemble averaging. Note that only the second indicator function contains τ as a variable and the second integral in Eq. (13)

$$\text{S.I.} \equiv \int_0^\infty d\tau e^{-s\tau} I(\bar{t}_{n_1+n_2} - t_1 < \tau < \bar{t}_{n_1+n_2+1} - t_1)$$

is rather trivial and it exhibits three behaviors: (i) S.I. = 0 for $\bar{t}_{n_1+n_2+1} - t_1 < 0$, (ii)

$$\text{S.I.} = \int_0^{\bar{t}_{n_1+n_2+1}-t_1} e^{-s\tau} d\tau = \frac{1 - e^{-s(\bar{t}_{n_1+n_2+1}-t_1)}}{s}$$

for $\bar{t}_{n_1+n_2} < t_1 < \bar{t}_{n_1+n_2+1}$ and (iii)

$$\begin{aligned} \text{S.I.} &= \int_{\bar{t}_{n_1+n_2}-t_1}^{\bar{t}_{n_1+n_2+1}-t_1} e^{-s\tau} d\tau \\ &= \frac{e^{-s(\bar{t}_{n_1+n_2}-t_1)} - e^{-s(\bar{t}_{n_1+n_2+1}-t_1)}}{s} \end{aligned}$$

for $t_1 < \bar{t}_{n_1+n_2}$. We note that condition (ii) and the condition $\bar{t}_{n_1} < t_1 < \bar{t}_{n_1+1}$ can hold simultaneously only when $n_2=0$, while condition (iii) and the condition $\bar{t}_{n_1} < t_1 < \bar{t}_{n_1+1}$ can only hold simultaneously if $n_2 \neq 0$. Using these behaviors we now get from Eq. (13)

$$\begin{aligned} P_{n_1, 0}(\lambda_1, s) &= \left\langle \int_{\bar{t}_{n_1}}^{\bar{t}_{n_1+1}} dt_1 e^{-\lambda_1 t_1} I(\bar{t}_{n_1} < t_1 < \bar{t}_{n_1+1}) \frac{1 - e^{-s(\bar{t}_{n_1+1}-t_1)}}{s} \right\rangle \\ &= \left\langle \frac{e^{-\bar{t}_{n_1}\lambda_1} - e^{-\lambda_1 \bar{t}_{n_1+1}}}{s\lambda_1} - \frac{e^{-s\bar{t}_{n_1+1}} e^{-(\lambda_1-s)\bar{t}_{n_1}} - e^{-(\lambda_1-s)\bar{t}_{n_1+1}}}{s(\lambda_1-s)} \right\rangle \end{aligned} \quad (14)$$

for $n_2 = 0$ and

$$\begin{aligned} P_{n_1, n_2}(\lambda_1, s) &= \left\langle \int_0^{\bar{t}_{n_1+n_2}} dt_1 e^{-\lambda_1 t_1} I(\bar{t}_{n_1} < t_1 < \bar{t}_{n_1+1}) \frac{e^{-s(\bar{t}_{n_1+n_2}-t_1)} - e^{-s(\bar{t}_{n_1+n_2+1}-t_1)}}{s} \right\rangle \\ &= \left\langle \frac{e^{-s\bar{t}_{n_1+n_2}} - e^{-s\bar{t}_{n_1+n_2+1}}}{s} \frac{e^{-(\lambda_1-s)\bar{t}_{n_1}} - e^{-(\lambda_1-s)\bar{t}_{n_1+1}}}{\lambda_1-s} \right\rangle \end{aligned} \quad (15)$$

for $n_2 \neq 0$.

Since waiting times are IID random variables, one has

$$\langle e^{-\bar{t}_{n_1}\lambda_1} \rangle = \psi^{n_1}(\lambda_1) \quad (16)$$

and

$$\langle e^{-s\bar{t}_{n_1+1}} e^{-(\lambda_1-s)\bar{t}_{n_1}} \rangle = \langle e^{-\lambda_1 \bar{t}_{n_1}} e^{-s(\bar{t}_{n_1+1}-\bar{t}_{n_1})} \rangle$$

$$= \psi^{n_1}(\lambda_1) \psi(s). \quad (17)$$

Similar expressions hold also for other terms in Eqs. (14) and (15). Here $\psi(\lambda_1)$ and $\psi(s)$ are Laplace transforms of the waiting time PDF $\psi(t_1)$ and $\psi(\tau)$ respectively. Using

these expressions we get:

$$P_{n_1,0}(\lambda_1, s) = \frac{\psi^{n_1}(\lambda_1)}{s} \left[\frac{1 - \psi(\lambda_1)}{\lambda_1} - \frac{\psi(s) - \psi(\lambda_1)}{\lambda_1 - s} \right] \quad (18)$$

and

$$P_{n_1,n_2}(\lambda_1, s) = \frac{\psi^{n_1}(\lambda_1) \psi^{n_2-1}(s)}{s} [1 - \psi(s)] \frac{[\psi(s) - \psi(\lambda_1)]}{(\lambda_1 - s)} \quad (19)$$

for $n_2 \geq 1$. Note that Eqs. (18,19) give the proper normalization since $\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} P_{\lambda_1,s}(n_1, n_2) = 1/(\lambda_1 s)$.

We now consider limiting behaviors of Eqs. (18,19). The probability of making no steps in the time interval $(t_1, t_1 + \tau)$ is given in double Laplace representation by

$$\sum_{n_1=0}^{\infty} P_{n_1,0}(\lambda_1, s) = \frac{1}{s\lambda_1} - \frac{\psi(s) - \psi(\lambda_1)}{s(\lambda_1 - s)[1 - \psi(\lambda_1)]}, \quad (20)$$

which was obtained previously [23]. Let t_f be the time between t_1 and the first jump event after t_1 : $t_f = \bar{t}_{n_1+1} - t_1$. The random variable t_f is sometimes called the forward recurrence time. Let its PDF be given by $h(t_f; t_1)$ depending on t_1 as a parameter. In double Laplace representation $t_1 \rightarrow \lambda_1$ and $t_f \rightarrow s$ one finds

[23, 24]

$$h(s; \lambda_1) = \frac{\psi(s) - \psi(\lambda_1)}{(\lambda_1 - s)[1 - \psi(\lambda_1)]}, \quad (21)$$

as follows from Eq. (20) by noting that the probability of making no jump in the time interval $(t_1, t_1 + \tau)$ is $1 - \int_0^\tau h(t_f; t_1) dt_f$. The probability of making $n_2 \geq 1$ jumps in $(t_1, t_1 + \tau)$ is

$$\sum_{n_1=0}^{\infty} P_{n_1,n_2}(\lambda_1, s) = h(s; \lambda_1) \frac{1 - \psi(s)}{s} \psi^{n_2-1}(s). \quad (22)$$

In the space of originals this equation corresponds to a convolution of the PDF of forward recurrence time t_f with the PDFs of the the following $n_2 - 1$ waiting times; the factor $[1 - \psi(s)]/s$ comes from the probability of not jumping between the last event in the sequence and the end of observation at t_2 .

IV. TWO-POINT CHARACTERISTIC FUNCTIONS

We are now able to find the characteristic function $g(k_1, \lambda_1; k, s)$, Eq. (11), using Eqs. (18,19):

$$\begin{aligned} g(k_1, \lambda_1; k, s) &= \left[\frac{1 - \psi(\lambda_1)}{\lambda_1 s} - \frac{\psi(s) - \psi(\lambda_1)}{s(\lambda_1 - s)} \right] \frac{1}{1 - \psi(\lambda_1) f(k_1)} \\ &+ \frac{f(k) [1 - \psi(s)] [\psi(s) - \psi(\lambda_1)]}{s(\lambda_1 - s)} \frac{1}{1 - \psi(\lambda_1) f(k_1)} \frac{1}{1 - \psi(s) f(k)} \\ &= \left[\frac{1}{\lambda_1 s} - \frac{h(s; \lambda_1)}{s} \right] \frac{1 - \psi(\lambda_1)}{1 - \psi(\lambda_1) f(k_1)} + \frac{1 - \psi(\lambda_1)}{1 - \psi(\lambda_1) f(k_1)} \frac{h(s; \lambda_1) f(k)}{s} \frac{1 - \psi(s)}{1 - \psi(s) f(k)}. \end{aligned} \quad (23)$$

This equation can be written in a more transparent way. First we recall the Montroll–Weiss equation. For a CTRW starting at time $t = 0$, the single point PDF $P_{\text{MW}}(x, t)$ of the particle being at site x at time t , is given in Laplace $t \rightarrow \lambda$ Fourier $x \rightarrow k$ space in terms of the Montroll–Weiss equation [1]

$$P_{\text{MW}}[\psi(\lambda), f(k)] = \frac{1 - \psi(\lambda)}{\lambda} \frac{1}{1 - f(k)\psi(\lambda)}. \quad (24)$$

The Montroll–Weiss equation explicitly assumes that the waiting time for the first step has the same PDF as all further waiting times. On the other hand, one may consider situations where the waiting time PDF for the first step, $\psi_1(t)$ differs from the PDFs of all other waiting times $\psi(t)$ [25]. The single point PDF describing this more general process [10, 11] called aging random walk [26] is denoted by $P_{\text{ARW}}(x, t)$. In the Laplace–Fourier

representation $x \rightarrow k, t \rightarrow \lambda$ one finds

$$P_{\text{ARW}}[\psi_1(\lambda), \psi(\lambda), f(k)] = \frac{1 - \psi_1(\lambda)}{\lambda} + \frac{\psi_1(\lambda) f(k)}{1 - f(k)\psi(\lambda)} \frac{1 - \psi(\lambda)}{\lambda}. \quad (25)$$

The aging random walk reduces to the Montroll–Weiss CTRW if $\psi_1(t) = \psi(t)$. Using Eqs. (24) and Eq.(25) one can rewrite Eq. (23) as

$$g(k_1, \lambda_1; k, s) = P_{\text{MW}}[\psi(\lambda_1), f(k_1)] P_{\text{ARW}}[\lambda_1 h(s; \lambda_1), \psi(s), f(k)]. \quad (26)$$

We see that the solution for g corresponds to a convolution of two PDFs, the one of the Montroll–Weiss CTRW and the one of the aging CTRW with the first waiting time PDF formally put to $\psi_1(\tau) = dh(\tau, t)/dt$. We note that the fact that the final characteristic function is a convolution and not a simple product of $P_{\text{MW}}(x_1, t_1)$ and

$P_{\text{ARW}}(\Delta, \tau; t_1)$, as found for simple Markovian diffusion, has to do with the correlations between x_1 and x_2 which in turn is related to the correlation between n_1 and n_2 . These arise through subtle correlations between the num-

ber of steps n_1 and the forward recurrence time.

After getting $g(k_1, \lambda_1; k, s)$ we can turn to the original characteristic function. Using Eqs. (6, 26) we find

$$p(k_1, \lambda_1; k_2, \lambda_2) = \sum_{i=1,2} P_{\text{MW}}[\psi(\lambda_1 + \lambda_2), f(k_1 + k_2)] P_{\text{ARW}}[(\lambda_1 + \lambda_2) h(\lambda_1 + \lambda_2; \lambda_i), \psi(\lambda_i), f(k_i)]. \quad (27)$$

One can check that if $k_1 = 0$ or $k_2 = 0$ (i.e. integrating the overall distribution over x_1 or x_2 respectively) we recover the Montroll–Weiss equation (24) for a one-point characteristic function, for example

$$p(k_1 = 0, \lambda_1; k_2, \lambda_2) = \frac{1}{\lambda_1 \lambda_2} \frac{1 - \psi(\lambda_2)}{1 - \psi(\lambda_2) f(k_2)} \quad (28)$$

as it should.

We now investigate the continuum limit of our main equation (27), corresponding to long t_1 and t_2 , using the standard long wave length (small k) and small frequency approximation [4]. We consider first the non-biased random walks with a finite second moment of jump lengths

$\langle \delta x^2 \rangle$, which means that for small k

$$f(k) \sim 1 - \frac{\langle \delta x^2 \rangle k^2}{2} + \dots \quad (29)$$

For $\lambda_1 \rightarrow 0$

$$\psi(\lambda_1) \sim 1 - A(\lambda_1)^\alpha \quad (30)$$

where $0 < \alpha \leq 1$ and $A > 0$. For the special case $\alpha = 1$, A is the mean waiting time. This case corresponds to asymptotically normal diffusion. If $\alpha < 1$ the mean time between jumps diverges, which leads to anomalous behaviors. In this limit

$$p(k_1, \lambda_1; k_2, \lambda_2) \sim \frac{(\lambda_1 + \lambda_2)^{\alpha-1}}{(\lambda_1 + \lambda_2)^\alpha + D_\alpha |k_1 + k_2|^2} \left\{ \sum_{i=1,2} \frac{(\lambda_1 + \lambda_2) (\lambda_i)^\alpha - \lambda_i (\lambda_1 + \lambda_2)^\alpha}{\lambda_1 \lambda_2 (\lambda_1 + \lambda_2)} + \right. \\ \left. + (\lambda_1 + \lambda_2) \frac{(\lambda_1 + \lambda_2)^\alpha - (\lambda_i)^\alpha}{\lambda_i (\lambda_1 + \lambda_2)^\alpha} \frac{(\lambda_1)^{\alpha-1}}{(\lambda_i)^\alpha + D_\alpha (k_i)^2} \right\} \quad (31)$$

where $\bar{1} = 2$ and $\bar{2} = 1$ and $D_\alpha = \langle \delta x^2 \rangle / (2A)$ is the fractional diffusion constant [8]. Previously Baule and Friedrich [18] wrote a multi-point fractional diffusion equation for $p(x_1, t_1; x_2, t_2)$, whose solution in Laplace-Fourier space is exactly Eq.(31).

V. CORRELATION FUNCTION FOR BIASED CTRW

We now consider the simplest correlation function

$$\langle x_1(\lambda_1) x_2(\lambda_2) \rangle = - \frac{\partial}{\partial k_1} \frac{\partial}{\partial k_2} p(k_1, \lambda_1; k_2, \lambda_2) |_{k_1=k_2=0}. \quad (32)$$

For a biased CTRW with finite variance of jump lengths, the small k expansion reads

$$f(k) \sim 1 + i \langle \delta x \rangle k - \frac{\langle \delta x^2 \rangle k^2}{2} + \dots, \quad (33)$$

where $\langle \delta x \rangle$ is the mean step length. Using Eqs. (27,32,33)

$$\langle x_1(\lambda_1) x_2(\lambda_2) \rangle = \frac{\langle \delta x^2 \rangle}{\lambda_1 \lambda_2 [1 - \psi(\lambda_1 + \lambda_2)]} \\ + \frac{2 \langle \delta x \rangle^2}{\lambda_1 \lambda_2 [1 - \psi(\lambda_1 + \lambda_2)]^2} \quad (34) \\ + \frac{\langle \delta x \rangle^2}{1 - \psi(\lambda_1 + \lambda_2)} \sum_{i=1,2} \frac{h(\lambda_1 + \lambda_2; \lambda_i)}{\lambda_i [1 - \psi(\lambda_i)]}.$$

Now we pass to the small λ_1 and λ_2 limit and consider the scaling limit of large t_1 and t_2 when their ratio is arbitrary, using Eqs. (30,34)

$$\langle x_1(\lambda_1) x_2(\lambda_2) \rangle \sim \frac{\langle \delta x^2 \rangle}{A \lambda_1 \lambda_2 (\lambda_1 + \lambda_2)^\alpha} +$$

$$+ \frac{\langle \delta x \rangle^2}{A^2} \frac{1/(\lambda_1)^\alpha + 1/(\lambda_2)^\alpha}{\lambda_1 \lambda_2 (\lambda_1 + \lambda_2)^\alpha}. \quad (35)$$

Laplace transform of Eq. (35) is found using the approach discussed in Appendix B of Ref.[17]. For $t_2 > t_1$ one has:

$$\begin{aligned} \langle x_1(t_1)x_2(t_2) \rangle &\sim \frac{\langle \delta x^2 \rangle}{A} \frac{(t_1)^\alpha}{\Gamma(1+\alpha)} + \\ &\frac{\langle \delta x \rangle^2}{A^2} \left[\frac{(t_1)^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{(t_1 t_2)^\alpha}{\Gamma(1+\alpha)^2} F\left(\alpha, -\alpha, \alpha+1; \frac{t_1}{t_2}\right) \right]. \end{aligned} \quad (36)$$

Taking $t_1 > t_2$ corresponds to simple interchange of the arguments. Here $F(a, b; c; z)$ is a hypergeometric function [27]. Let us check limiting behaviors of the correlation function Eq. (36). For an unbiased process $\langle \delta x \rangle = 0$ the first term on the right hand side of equation (36) is the only non-vanishing term

$$\langle x_1(t_1)x_2(t_2) \rangle \sim \frac{\langle \delta x^2 \rangle}{A} \frac{(t_1)^\alpha}{\Gamma(1+\alpha)} \quad (37)$$

which represents the dispersion of the walker's positions at time t_1 . On the other hand, for the biased CTRW $\langle \delta x \rangle \neq 0$ the leading behavior is given by the second line in Eq. (36) and we may neglect the first term. Several limiting situations can be used as benchmarks. For $t_2 \gg t_1$ one has $F(\alpha, -\alpha; \alpha+1; x) \sim 1 - \alpha^2 x / (1+\alpha) + O(x^2)$ so that

$$\langle x_1(t_1)x_2(t_2) \rangle \sim \langle x_1(t_1) \rangle \langle x_2(t_2) \rangle \quad (38)$$

with

$$\begin{aligned} \langle x_1(t_1) \rangle &\sim \frac{\langle \delta x \rangle}{A} \frac{(t_1)^\alpha}{\Gamma(1+\alpha)} \\ \langle x_1(t_2) \rangle &\sim \frac{\langle \delta x \rangle}{A} \frac{(t_2)^\alpha}{\Gamma(1+\alpha)}, \end{aligned} \quad (39)$$

which proves the decoupling of correlations for $t_2 \gg t_1$. In the opposite limit of $t_2 \rightarrow t_1$ one uses Eq.(15.3.6) of Ref.([27]) and $F(a, b; c; 0) = 1$ to get $F(\alpha, -\alpha; \alpha+1; 1) = [\Gamma(\alpha+1)]^2 / \Gamma(2\alpha+1)$ and to obtain

$$\lim_{t_2 \rightarrow t_1} \langle x_1(t_1)x_2(t_2) \rangle \sim \frac{\langle \delta x^2 \rangle}{A} \frac{(t_1)^\alpha}{\Gamma(1+\alpha)} + \frac{2\langle \delta x \rangle^2 (t_1)^{2\alpha}}{\Gamma(1+2\alpha)A^2}, \quad (40)$$

which is the mean square displacement in the biased CTRW [28], as expected.

Another limit is the Markovian case $\alpha = 1$, for which $F(1, -1; 2; x) = 1 - x/2$. In this case the decoupling,

Eq.(38), is valid at all times. The fact that for $\alpha < 1$ Eq.(38) holds only for $t_2 \gg t_1$, i.e. the existence of non-trivial correlations between x_1 and x_2 , has to do with the correlations between the number of steps before and after the first observation time t_1 which we discussed in Sec. IV. Thus, if relatively few jumps take place during the time interval $(0, t_1)$, i.e. $n_1 \ll \langle n_1 \rangle$, the typical displacement x_1 is inevitably small, and then the particle is likely to be effectively trapped at its position at t_1 for a very long time which is of the order of t_1 . In this case also the forward recurrence time is long in statistical sense. This implies that also n_2 is going to be relatively small, since the particle will likely wait for a long time for its first step after t_1 , which leads to small absolute values of x_2 as well. Hence correlations for $\alpha < 1$ are built even when t_1 and t_2 are very long.

VI. DISCUSSION

The Montroll–Weiss equation (24) expresses the characteristic function of the CTRW in terms of Laplace and Fourier transforms of the PDFs of the waiting times and jump lengths. Similarly, Eq. (27) gives the two dimensional characteristic function of the CTRW process. From this equation we may derive two dimensional correlation functions for the CTRW process, for example we considered the biased CTRW. We showed that the two dimensional characteristic function depends on the probability of n_1 renewals in $(0, t_1)$ and n_2 renewals in (t_1, t_2) (for $t_2 > t_1$) and that these numbers of steps are correlated. For characteristic functions of order N higher than two one would have to calculate renewal statistics in N intervals. In principle this calculation can be performed using the same technique we used here for example to calculate $P_{n_1, n_2, n_3}(t_1, t_2, t_3)$ etc.

The two dimensional characteristic function is shown to be related to the Montroll–Weiss and aging CTRW single point characteristic functions. Thus even though the process is non-Markovian information on one dimensional characteristic functions is sufficient to find the two dimensional characteristic function. This simplification is obviously related to the renewal property of the underlying random walk. Finally, starting with the CTRW model we derived the solution of the multi-point fractional diffusion equation [18], in Fourier-Laplace space, thus giving further justification for this new equation.

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